SUMS OF PRODUCTS OF THE TERMS OF THE GENERALIZED LUCAS SEQUENCE \{V_{kn}\}

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Abstract
In this study we consider the generalized Lucas sequence \{V_n\} with indices in arithmetic progression. We also compute the sums of products of the terms of the Lucas sequence \{V_{kn}\} for positive odd integers \(k\).

Keywords: Second order linear recurrence, Fibonomial coefficients.


1. Introduction
The binary linear recurrence \(W_n = W_n(a, b; p, q)\) is defined as follows for \(n > 1\),
\[
W_n = pW_{n-1} + qW_{n-2},
\]
where \(W_0 = a, W_1 = b\).

The Binet formula for \(\{W_n\}\) is
\[
W_n = A\alpha^n + B\beta^n,
\]
where \(A = \frac{b-a\beta}{\alpha-\beta}, B = \frac{a\alpha-b}{\alpha-\beta}\) and \(\alpha, \beta = \left(p \pm \sqrt{p^2 + 4q}\right)/2\).

For \(n > 1\) and a fixed positive integer \(k\), the terms of \(\{W_{kn}\}\) satisfy the recursion [6, 7]:
\[
W_{kn} = V_k W_{k(n-1)} - (-q)^k W_{k(n-2)},
\]
where \( V_k = \alpha^k + \beta^k \). Specifically, define the generalized Fibonacci \( \{ U_n \} \) and Lucas \( \{ V_n \} \) sequences as \( U_n = W_n (0, 1; p, 1) \), \( V_n = W_n (2, p; p, 1) \), respectively. Thus:

\[
(1.2) \quad U_{kn} = V_k U_{k(n-1)} - (-1)^k U_{k(n-2)},
\]

\[
(1.3) \quad V_{kn} = V_k V_{k(n-1)} - (-1)^k V_{k(n-2)}.
\]

The Fibonomial coefficients \( \binom{n}{m}_F \) are defined by the relation

\[
\binom{n}{m}_F = \frac{F_1 F_2 \cdots F_n}{(F_1 F_2 \cdots F_{n-m}) (F_1 F_2 \cdots F_m)}.
\]

for \( n \geq m \geq 1 \), with \( \binom{n}{0}_F = \binom{n}{n}_F = 1 \), where \( F_n \) is the \( n \)th Fibonacci number. These coefficients satisfy the relation:

\[
\binom{n}{m}_F = F_{m+1} \binom{n-1}{m} + F_{n-m+1} \binom{n-1}{m-1}.
\]

Hoggatt [4] defined the following generalization by taking \( F_{kn} \) instead of \( F_n \) for a fixed positive integer \( k \):

\[
\binom{n}{m}_{F_k} = \frac{F_k F_{2k} \cdots F_{kn}}{(F_k F_{2k} \cdots F_{k(n-m)}) (F_k F_{2k} \cdots F_{km})}.
\]

Jarden and Motzkin were the first to study the generalized Fibonomial coefficients formed by terms of the sequence \( \{ U_n \} \) as follows: for \( n \geq m \geq 1 \),

\[
\binom{n}{m}_U = \frac{U_1 U_2 \cdots U_n}{(U_1 U_2 \cdots U_{n-m}) (U_1 U_2 \cdots U_m)},
\]

with \( \binom{n}{0}_U = \binom{n}{n}_U = 1 \).

When \( p = 1 \), the generalized Fibonomial coefficients \( \binom{n}{m}_{U_k} \) are reduced to the Fibonomial coefficients \( \binom{n}{m}_F \).

By taking \( U_{kn} \) instead of \( U_n \) for a fixed positive integer \( k \), one can get

\[
\binom{n}{m}_{U_k} = \frac{U_k U_{2k} \cdots U_{kn}}{(U_k U_{2k} \cdots U_{k(n-m)}) (U_k U_{2k} \cdots U_{km})}.
\]

These coefficients satisfy the relations

\[
\binom{n}{m}_{U_k} = U_{km+1} \binom{n-1}{m}_{U_k} + U_{k(n-m)-1} \binom{n-1}{m-1}_{U_k}
\]

and

\[
\binom{n}{m}_{U_k} = U_{km-1} \binom{n-1}{m}_{U_k} + U_{k(n-m)+1} \binom{n-1}{m-1}_{U_k}.
\]

Golomb [3] found the generating function for the numbers \( F_n^m \), and this result started the effort to find a recurrences or closed form for the generating function

\[
f_m (x) = \sum_{n=0}^{\infty} F_n^m x^n
\]

of the \( m \)th powers of the Fibonacci numbers.

In [8], Riordan found the general recurrence relation for \( f_m (x) \) (see also [2]). Carlitz [1] and Horadam [5] generalized the result of Riordan and found similar recurrences for the generating functions of different types of generalized Fibonacci numbers. They also
found a closed form for the polynomial $N_m(x)$ in the numerator, and the polynomial $D_m(x)$ in the denominator of the generating functions. As a special case of Horadam’s result, it is possible to get the following formula for the generating function of odd integer powers of the Fibonacci numbers:

\[
(1.4) \quad f_m(x) = \frac{\sum_{i=0}^{m-1} (-1)^{i+1} \left\lfloor \frac{m+1}{j} \right\rfloor \cdot F_{i-j}^m x^j}{\sum_{i=0}^{m+1} (-1)^{i+1} \left\lfloor \frac{m+1}{j} \right\rfloor \cdot F_{i-j}^m x^j}.
\]

In [13], applying Carlitz’s approach, Shannon obtained some special results for the numerator and the denominator in the expression of the generating function $f_m(x)$. Using the $q$-analogue of the terminating binomial theorem, he obtained the relation

\[
\prod_{i=0}^{m} \left(1 - q^i x\right) = \sum_{i=0}^{m+1} (-1)^i q^{i(q-1)} \cdot \binom{m+1}{i} x^i,
\]

where $\binom{m}{i} = \frac{(1-q^{m+1})(1-q^{m+2}) \cdots (1-q^{m+i})}{(1-q^i)(1-q^{i+1}) \cdots (1-q^m)}$ is the Gaussian $q$-binomial coefficient for $i \geq 1$, any complex numbers $q, x$ and any positive integer $m$ with $\binom{m}{0} = 1$. Replacing $q$ by $\beta/\alpha$ and $x$ by $x^m$, one can get

\[
\prod_{i=0}^{m} \left(1 - \alpha^{m-i} \beta^i x\right) = \sum_{i=0}^{m+1} (-1)^i \left(\frac{m+1}{i}\right) \cdot \binom{m+1}{i} \cdot x^i.\]

It is easy to obtain for any odd integer $m$ that

\[
(1.5) \quad f_m(x) = 5^{-m-1} \sum_{j=0}^{m-1} \binom{m}{j} \frac{F_{m-2j} x}{1 - (-1)^j L_{m-2j} x - x^2},
\]

after simplifications of one of Shannon’s results. Seibert and Trojovsky [11] gave certain generalizations of the well-known formulas for the Fibonacci and Lucas numbers. For example,

\[
\sum_{i=0}^{n} (-1)^i L_{m-2i} = 2F_{n+1}.
\]

For odd positive integer $m$, the authors concentrated on the sums

\[
\sum_{i_1=0}^{m/2} \sum_{i_2=1}^{m/2} \cdots \sum_{i_n=1}^{m/2} (-1)^{i_1+i_2+\cdots+i_n} \prod_{j=1}^{n} L_{m-2i_j}.
\]

Combining (1.4) and (1.5), they gave some new results about these sums with the help of the Fibonomial coefficients.

In [12], for arbitrary positive integer $m$, they gave some new results about these sums with the help of the Fibonomial coefficients.

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We consider the generalized Lucas sequence $\{V_n\}$ with indices in arithmetic progression, and then compute the sums of products of terms of the sequence $\{V_{kn}\}$ for a positive odd integer $k$. \[\begin{align*}
\text{(1.6) } & \sum_{i_1=0}^{m/2} \sum_{i_2=1}^{m/2} \cdots \sum_{i_n=1}^{m/2} (-1)^{i_1+i_2+\cdots+i_n} \prod_{j=1}^{n} L_{m-2i_j},
\end{align*}\]
2. Some identities including the terms of \( \{U_{kn}\} \) and \( \{V_{kn}\} \)

We shall give some results for later use. Throughout this study, we will denote \( W_n(a,b,p,1) \) by \( H_n \).

2.1. Lemma. For \( m, n > 0 \),

\[
\begin{align*}
(2.1) \quad V_{k(m+n)} + V_{k(m-n)} &= \begin{cases} 
\frac{\sqrt{5}+1}{2}U_{k}U_{kn} & \text{if } n \text{ is odd}, \\
V_{km}V_{kn} & \text{if } n \text{ is even,}
\end{cases} \\
(2.2) \quad V_{k(m+n)} - V_{k(m-n)} &= \begin{cases} 
V_{km}V_{kn} & \text{if } n \text{ is odd,} \\
\frac{\sqrt{5}+1}{2}U_{km}U_{kn} & \text{if } n \text{ is even,}
\end{cases} \\
(2.3) \quad U_{k(m+n)} + U_{k(m-n)} &= \begin{cases} 
V_{km}U_{kn} & \text{if } n \text{ is odd,} \\
U_{km}V_{kn} & \text{if } n \text{ is even,}
\end{cases} \\
(2.4) \quad U_{k(m+n)} - U_{k(m-n)} &= \begin{cases} 
U_{km}V_{kn} & \text{if } n \text{ is odd,} \\
V_{km}U_{kn} & \text{if } n \text{ is even,}
\end{cases} \\
(2.5) \quad V_{k(m-2n)}V_{k(2m-2n)-3} &= \begin{cases} 
V_{3k(m-2n-1)} - V_{k(m-2n-3)} & \text{if } m \text{ is odd,} \\
V_{3k(m-2n-1)} + V_{k(m-2n-3)} & \text{if } m \text{ is even,}
\end{cases} \\
(2.6) \quad V_{k(m-2n)}V_{k(m-2n-3)} &= \begin{cases} 
U_{k(2m-4n-1)} - U_{k} & \text{if } m \text{ is odd,} \\
U_{k(2m-4n-1)} + U_{k} & \text{if } m \text{ is even,}
\end{cases} \\
(2.7) \quad V_{3k(m+1)} + V_{3k(m-1)} &= (V_k^2 + 4)U_{3km}U_{3k}, \\
(2.8) \quad V_{kn}U_{k(n-1)} - U_{k(2n-1)} &= (-1)^{kn}U_k, \\
(2.9) \quad U_{k(m+n)} = U_{km}V_{kn} + (-1)^{n+1}U_{k(m-n)}, \\
(2.10) \quad U_{k(m+n)}U_{k(m+t)} - U_{km}U_{k(m+t+n)} &= (-1)^{m}U_{kn}U_{kt}.
\end{align*}
\]

Proof. The proof follows by the Binet formulas for \( \{U_{kn}\} \) and \( \{V_{kn}\} \). □

2.2. Theorem. For any integers \( r, c, d \) with \( c \neq 0 \) and \( n \geq 0 \),

\[
\begin{align*}
i) \quad (2.11) \quad \sum_{i=r}^{n} H_{k(ci+d)} &= [H_{k(cr+d)} - 2^{c}H_{k(c(r+1)+d)} - (-1)^{c+1}H_{k(c(r-1)+d)} + (-1)^{c}H_{k(cn+d)}] + \frac{1}{V_{kc} + (-1)^{c}}, \\
&+ \sum_{i=r}^{n} (-1)^{c}H_{k(ci+d)} = [(-1)^{c}H_{k(cr+d)} + (-1)^{n}H_{k(c(n+1)+d)} + (-1)^{c+1}H_{k(c(r+1)+d)} + (-1)^{c+n}H_{k(cn+d)}] + \frac{1}{V_{kc} + (-1)^{c}}.
\end{align*}
\]

Proof. i) By the Binet formula for \( \{H_{kn}\} \), we have

\[
\sum_{i=r}^{n} H_{k(ci+d)} = H_{k(cr+d)} + H_{k(c(r+1)+d)} + \cdots + H_{k(cn+d)}
\]
2.3. Theorem. $\alpha \beta$

\[= A\alpha^k + B\beta^k + A\alpha^{k+1} + B\beta^{k+1} + \cdots + A\alpha^{k+n} + B\beta^{k+n} + A\beta^k + B\alpha^k \]

\[= A\alpha^k + B\beta^k \left(1 + \alpha^{kc} + \alpha^{2kc} + \cdots + \alpha^{kc(n-r)}\right) + B\beta^k \left(1 + \beta^{kc} + \beta^{2kc} + \cdots + \beta^{kc(n-r)}\right)\]

\[= A\alpha^k + B\beta^k \left(1 - \frac{\alpha^{kc(n-r+1)}}{1 - \alpha^{kc}} + B\beta^k \frac{1 - \beta^{kc(n-r+1)}}{1 - \beta^{kc}}\right),\]

which, by $(\alpha\beta)^k = -1$ gives us

\[
\sum_{i=r}^{n} H_i^{(c+i+d)} = \left[ A\alpha^k + B\beta^k \right] - \left( A\alpha^k + B\beta^k \right) \]

\[= (-1)^c \left( A\alpha^k + B\beta^k \right) \]

\[= (H_k^{(c+d)} - H_k^{(c(n+1)+d)} - (-1)^c H_k^{(c(r-1)+d)} + (-1)^c H_k^{(c(n+d))} \right) / (1 - V_k + (-1)^c).\]

Thus the proof of (i) is complete. The equation (2.12) can be similarly proven. □

2.3. Theorem. For any integers $r, c (c \neq 0), d$ and $n \geq 0$,

\[
\sum_{i=r}^{n} i H_i^{(c+i+d)} = \left( n H_k^{(c(n+2)+d)} - (n + 1 + 2n (-1)^c) H_k^{(c(n+1)+d)} \right) \]

\[+ (n + 1) (-1)^c H_k^{(c(n+d))} - (n + 1) H_k^{(c(n-1)+d)} - (r + 1 + 2r (-1)^c) H_k^{(c(r-1)+d)} \]

\[+ r H_k^{(c(r-2)+d)} + (r + 2 (r - 1) (-1)^c) H_k^{(c(r+d))} \]

\[= \left( (n (-1)^{n+1} - 2 (n + 1) (-1)^{c+n}) H_k^{(c(n+d))} \right) \]

\[= (n (n+1) (-1)^n) H_k^{(c(n-1)+d)} \]

\[+ (2n (-1)^{c+n+1} - (n + 1) (-1)^n) H_k^{(c(n+1)+d)} \]

\[+ n (-1)^{n+1} H_k^{(c(n+2)+d)} - \left( (r - 1) (-1)^r - 2r (-1)^{c+r-1} \right) H_k^{(c(r-1)+d)} \]

\[+ (r (-1)^{r-1} - 2 (r - 1) (-1)^{r+1}) H_k^{(c(r-2)+d)} \]

\[= (-r - 1) (-1)^r H_k^{(c(r+1)+d)} \right) / (1 + V_k + (-1)^c)^2.\]
Proof. Theorem 2.3 is proved by considering
\[ \sum_{i=r}^{n} ix^{i-1} = \frac{n x^{n+1} - (n + 1) x^n - (r - 1) x^r + rx^{r-1}}{(x-1)^2}, \]
and using the Binet formula. \( \square \)

2.4. Corollary. For an odd positive integer \( m \),

\begin{align}
(2.15) & \sum_{i=j+1}^{m-1} U_{k(2m-4i-1)} = \frac{U_k}{V_k (V_k^2 + 4)} \left[ V_{k(2m-4j-3)} + V_k \right], \\
(2.16) & \sum_{i=r}^{m-1} (-1)^i V_{k(m-2i)} = (-1)^r \frac{U_{k(m-2r+1)}}{U_k}, \\
(2.17) & \sum_{i=0}^{m-1} (-1)^i V_{k(3(m-2i-1))} = (-1)^{\frac{m-1}{2}} + \frac{U_{3km} V_k}{(V_k^2 + 1)^2}, \\
(2.18) & \sum_{i=0}^{m-1} (-1)^{i-1} iV_{k(m-2i)} = \frac{(-1)^{\frac{m-1}{2}} V_k - V_{km}}{V_k^2 + 4},
\end{align}

Proof. Substituting \( n = \frac{m-1}{2} \), \( c = -4 \), \( d = 2m - 1 \) and \( r = j + 1 \) in (2.11) gives us
\[ \sum_{i=j+1}^{m-1} U_{k(2m-4i-1)} = \frac{-U_{k(2m-4j-5)} + U_{k(2m-4j-1)} + U_{3km} - U_k}{V_k (V_k^2 + 4)}, \]
which, by (2.4), is equal to
\[ \frac{U_{2k} V_{k(2m-4j-3)} + U_k V_k^2}{V_k (V_k^2 + 4)} = \frac{U_k}{V_k (V_k^2 + 4)} \left[ V_{k(2m-4j-3)} + V_k \right], \]
as claimed in (2.15). Using Lemma 2.1, the identities (2.16)-(2.18) can be proved in a similar way as in the proof of (2.15). \( \square \)

2.5. Corollary. For an even positive integer \( m \),

\begin{align}
(2.19) & \sum_{i=j+1}^{m-2} U_{k(2m-4i-1)} = \frac{U_k}{V_k (V_k^2 + 4)} \left[ V_{k(2m-4j-3)} - V_k \right], \\
(2.20) & \sum_{i=r}^{m-2} (-1)^i V_{k(m-2i)} = (-1)^r \frac{U_{k(m-2r+1)}}{U_k} - (-1)^{\frac{m}{2}}, \\
(2.21) & \sum_{i=0}^{m-2} (-1)^i V_{k(3(m-2i-1))} = \frac{U_{3km}}{U_k (V_k^2 + 1)}, \\
(2.22) & \sum_{i=0}^{m-2} (-1)^{i-1} iV_{k(m-2i)} = (-1)^{\frac{m}{2}} \frac{m}{2} + \frac{V_{km} - 2 (-1)^{\frac{m}{2}}}{V_k^2 + 4}. \quad \square
\end{align}
3. Sums of products of the terms of \( \{ V_{kn} \} \)

Define the sequence \( \{ S^k_n(m) \}_{n=0}^\infty \) in the following way: for \( m > 0 \)

\[
3.1 \quad S^k_0(m) = U_k, \quad S^k_1(m) = \sum_{i_1=0}^{\frac{m-1}{2}} (-1)^{i_1} V_{k(m-2i_1)}
\]

and for \( n > 1 \),

\[
3.2 \quad S^k_n(m) = \sum_{i_2=0}^{\frac{m-1}{2}} \sum_{i_1=i_2+1}^{\frac{m-1}{2}} \cdots \sum_{i_n=0}^{\frac{m-1}{2}} (-1)^{i_1+i_2+\cdots+i_n} \prod_{j=1}^{n} V_{k(m-2i_j)}.
\]

Throughout this section we shall frequently follow the organization of the work [11] while giving our results.

3.1. Theorem. For an odd positive integer \( m \),

\[
S^k_1(m) = \sum_{i_1=0}^{\frac{m-1}{2}} (-1)^{i_1} V_{k(m-2i_1)} = \frac{U_{k(m+1)}}{U_k},
\]

\[
S^k_2(m) = \sum_{i_2=0}^{\frac{m-1}{2}} \sum_{i_1=i_2+1}^{\frac{m-1}{2}} (-1)^{i_1+i_2} V_{k(m-2i_1)} V_{k(m-2i_2)} = \frac{m+1}{2} - \frac{1}{U_k U_{2k} U_{3k}} U_{km} U_{k(m+1)}.
\]

and

\[
S^k_3(m) = \sum_{i_3=0}^{\frac{m-1}{2}} \sum_{i_2=i_3+1}^{\frac{m-1}{2}} \sum_{i_1=i_2+1}^{\frac{m-1}{2}} (-1)^{i_1+i_2+i_3} V_{k(m-2i_1)} V_{k(m-2i_2)} V_{k(m-2i_3)} = \frac{U_{k(m+1)}}{U_k} - \frac{1}{U_k U_{2k} U_{3k}} U_{km} U_{k(m+1)} U_{k(m-1)}.
\]

Proof. i) Substituting \( n = \frac{m-1}{2} \), \( c = -2 \), \( d = m \) and \( r = 0 \) in (2.12), we get

\[
S^k_1(m) = \sum_{i_1=0}^{\frac{m-1}{2}} (-1)^{i_1} V_{k(m-2i_1)} = \frac{V_{km} + (-1)^{\frac{m-1}{2}} V_k + V_{k(m+2)} + (-1)^{\frac{m-1}{2}} V_k}{(V_k^2 + 4)}.
\]

Since \( V_{-k} = V_k \) and by (2.1), we get

\[
S^k_1(m) = \frac{V_{km} + V_{k(m+2)}}{(V_k^2 + 4)} = \sum_{i_1=0}^{\frac{m-1}{2}} (-1)^{i_1} V_{k(m-2i_1)} = \frac{U_{k(m+1)}}{U_k}.
\]
From (2.15), we have

\[ S_2^k (m) = \sum_{i_2=0}^{m-1} \sum_{i_1=i_2+1}^{m-1} (-1)^{i_1+i_2} V_k (m-2i_2) V_k (m-2i_1) \]

Taking \( j = -1 \) in (2.15), we get

\[ \sum_{i=0}^{m-1} U_k (2m-4i-1) = \frac{U_k (m+1) U_k m}{U_{2k}} \]

Then

\[ S_2^k (m) = \frac{m+1}{2} - \frac{1}{U_k U_{2k}} U_k m U_k (m+1) . \]

iii) Using (2.16) and (2.6), we write

\[ S_3^k (m) = \sum_{i_3=0}^{m-1} \sum_{i_2=i_3+1}^{m-1} \sum_{i_1=i_2+1}^{m-1} (-1)^{i_1+i_2+i_3} V_k (m-2i_3) V_k (m-2i_2) V_k (m-2i_1) \]

\[ = \frac{1}{U_k} \sum_{i_3=0}^{m-1} \sum_{i_2=i_3+1}^{m-1} (-1)^{i_3+1} V_k (m-2i_3) V_k (m-2i_2) U_k (m-2i_1-1) \]

\[ = \frac{1}{U_k} \sum_{i_3=0}^{m-1} \sum_{i_2=i_3+1}^{m-1} (-1)^{i_3+1} V_k (m-2i_3) (U_k (2m-4i_2-1) - U_k) \]

\[ = \frac{1}{U_k} \sum_{i_3=0}^{m-1} (-1)^{i_3+1} V_k (m-2i_3) \sum_{i_2=i_3+1}^{m-1} (U_k (2m-4i_2-1) - U_k) . \]

From (2.15), we have

\[ S_3^k (m) = \frac{1}{U_k} \sum_{i_3=0}^{m-1} (-1)^{i_3+1} V_k (m-2i_3) \]

\[ \times \left( \frac{U_k}{V_k (V_k^2 + 4)} (V_k (2m-4i_3-3) + V_k) + \left( i_3 - \frac{m-1}{2} \right) U_k \right) \]

\[ = \frac{1}{U_k} \sum_{i_3=0}^{m-1} (-1)^{i_3} V_k (m-2i_3) \]

\[ \times \left( - \frac{U_k}{V_k (V_k^2 + 4)} (V_k (2m-4i_3-3) + V_k) + \left( \frac{m-1}{2} - i_3 \right) U_k \right) . \]
From (2.16)-(2.18) and (2.1), we get

\[
\text{Proof.}
\]

The proof is similar to the proof of Theorem 3.1.

\[S_{3}^k(m) = \sum_{i_{3}=0}^{m-1} (-1)^{i_{3}} V_{k(m-2i_{3})} V_{k(2m-4i_{3}-3)} + \left( \frac{m-1}{2} - \frac{1}{V_{k}^{2} + 4} \right) \sum_{i_{3}=0}^{m-1} (-1)^{i_{3}} V_{k(m-2i_{3})} - \sum_{i_{3}=0}^{m-1} (-1)^{i_{3}} i_{3} V_{k(m-2i_{3})}.\]

Thus we have the conclusion. \(\square\)

### 3.2. Theorem

For even positive integer \(m\),

\[
S_{1}^k(m) = \sum_{i_{1}=0}^{m-2} (-1)^{i_{1}} V_{k(m-2i_{1})} = \frac{U_{k(m+1)}}{U_{k}} - (-1)^{\frac{m-2}{2}},
\]

\[
S_{2}^k(m) = \sum_{i_{2}=0}^{m-2} \sum_{i_{1}=i_{2}+1}^{m-2} (-1)^{i_{1}+i_{2}} V_{k(m-2i_{2})} V_{k(m-2i_{1})} = \frac{m-2}{2} + (-1)^{\frac{m-2}{2}} \frac{U_{k(m+1)}}{U_{k}} + \frac{U_{km} U_{k(m+1)}}{U_{k} U_{2k}},
\]

\[
S_{3}^k(m) = \sum_{i_{3}=0}^{m-2} \sum_{i_{2}=i_{3}+1}^{m-2} \sum_{i_{1}=i_{2}+1}^{m-2} (-1)^{i_{1}+i_{2}+i_{3}} V_{k(m-2i_{3})} V_{k(m-2i_{2})} V_{k(m-2i_{1})} = \left( \frac{m-4}{2} \right) (-1)^{\frac{m-2}{2}} \frac{U_{k(m+1)}}{U_{k}} + \frac{U_{km} U_{k(m+1)}}{U_{k} U_{2k} U_{3k}} \left( (-1)^{\frac{m-2}{2}} - \frac{U_{k(m+1)}}{U_{k} U_{2k} U_{3k}} \right).
\]

**Proof.** The proof is similar to the proof of Theorem 3.1. \(\square\)
In [14], Stanica gave the generating function for powers of the terms of the sequence \( \{W_n\} \), \( W(m, x) = \sum_{i=0}^{\infty} W^i_{m} x^i \) as follows:

### 3.3. Theorem

For \( n \geq 0 \) and odd positive integer \( m \),

\[
W(m, x) = \sum_{i=0}^{m-1} (-AB)^i \binom{m}{i} \times \frac{A^{m-2i} - B^{m-2i} + (-q)^i (B^{m-2i} \alpha^{m-2i} - A^{m-2i} \beta^{m-2i}) x}{1 - (-q)^i V_{m-2i} x - q^m x^2},
\]

and for even positive integer \( m \),

\[
W(m, x) = \sum_{i=0}^{m-1} (-AB)^i \binom{m}{i} \times \frac{B^{m-2i} + A^{m-2i} - (-q)^i (B^{m-2i} \alpha^{m-2i} + A^{m-2i} \beta^{m-2i}) x}{1 - (-q)^i V_{m-2i} x + q^m x^2} + \left( \frac{m}{2} \right) (AB)^{m/2} \text{.}
\]

### 3.4. Theorem

For \( n \geq 0 \) and an odd positive integer \( m \),

\[
S_n^k (m) = \sum_{i=0}^{\left\lfloor \frac{m+1}{2} \right\rfloor} (-1)^{\left\lfloor \frac{m}{2} \right\rfloor} \binom{m+1}{n-2i} U_k \theta (i, m, n) \binom{m+1}{i} \text{,}
\]

and for an even positive integer \( m \),

\[
S_n^k (m) = \sum_{i=0}^{\frac{n-2i}{2}} \sum_{j=0}^{\left\lfloor \frac{m+1}{2} \right\rfloor} (-1)^{i+n(\left\lfloor \frac{m+1}{2} \right\rfloor + j)} \binom{m+1}{j} U_k \theta (i, m, n) \binom{m+1}{i} \text{,}
\]

where \( \theta (i, m, n) = \left( \left\lfloor \frac{m+1}{2} \right\rfloor - n + i \right) + \left( \left\lfloor \frac{m+1}{2} \right\rfloor - n + i - 1 \right) \).

**Proof.** We give the proof for an odd integer \( m \). From (3.3), we write

\[
U_{kn} (m, x) = \frac{U_k}{\sqrt{k + 4}} \sum_{j=0}^{\frac{m-1}{2}} \binom{m}{j} \frac{U_k (m-2j) x}{1 - (-1)^j V_{k(m-2j)} x - x^2}.
\]

Relation (3.6), which hold for odd \( m \), leads to

\[
D_{m+1}^k (x) = \prod_{j=0}^{\frac{m-1}{2}} \left( 1 - (-1)^j V_{k(m-2j)} x - x^2 \right) = \sum_{i=0}^{m+1} d_{m+1,i} x^i,
\]

where \( d_{m+1,i} = (-1)^{\frac{(i+1)(i+2)}{2}} \left[ \binom{m+1}{i} U_k \right] \). After multiplication of all the factors in \( D_{m+1}^k (x) \), we obtain

\[
\begin{align*}
    d_{m+1,0} &= S_0^k (m) \text{,} & d_{m+1,i} &= \sum_{l=0}^{\left\lfloor \frac{m+1}{2} \right\rfloor} \left( \binom{m+1}{i} - (i - 2l) \right) (-1)^{i+l} S_{k-2l}^k (m) \text{.}
\end{align*}
\]
where \( i = 1, 2, \ldots, m + 1 \). Since \( \binom{n}{m} = (-1)^m \binom{m-n-1}{m} \), we rewrite the last identity for \( n > 0 \) as follows:

\[
d_{m+1,2n-1} = - \sum_{i=1}^{n} \left( \frac{n+i-m+1}{n-i} \right) S^k_{2i-1}(m)
\]

\[
d_{m+1,2(n-1)} = \sum_{i=1}^{n} \left( \frac{n+i-m+1}{n-i} \right) S^k_{2i-1}(m).
\]

By the binomial inversion theorem (for more details, see [10]),

\[
a_n = \sum_{i=1}^{n} \left( \frac{n+i+r}{n-i} \right) b_i
\]

holds if and only if

\[
b_n = \sum_{i=1}^{n} (-1)^{i+n} \left( \frac{2n+r}{n-i} - \frac{2n+r}{n-i-1} \right) a_i,
\]

where \( r \) is any integer. After this, by taking \( a_n = c_{m+1,2n-1}, b_i = -S^k_{2i-1} \) and \( r = \frac{m+5}{2} \) in (3.7), we obtain

\[
S^k_{2n-1}(m) = \sum_{i=1}^{n} (-1)^{i+n+1} \left[ \frac{2n-m+5}{n-i} - \frac{2n-m}{n-i-1} \right] d_{m+1,2i-1}

= \sum_{i=1}^{n} (-1) \left[ \frac{n-i+m+5}{n-i} - \frac{n-i+1}{n-i-1} \right] d_{m+1,2i-1}

(3.8)

= \sum_{i=1}^{n} (-1)^{(2i-1)i+1} \left[ \frac{n-i+m+5}{n-i} - \frac{n-i+1}{n-i-1} \right]

\times \binom{m+2}{2i-1}_U_k.

Similarly if we take \( a_n = c_{m+1,2(n-1)}, b_i = -S^k_{2i-1} \) and \( r = \frac{m+5}{2} \) in (3.7), then we get

\[
S^k_{2(n-1)}(m) = \sum_{i=1}^{n} (-1)^{(2i-1)i+1} \left[ \frac{n-i+m+5}{n-i} - \frac{n-i+1}{n-i-1} \right]

\times \binom{m+1}{2i-1}_U_k.

(3.9)

Combining (3.8) and (3.9), we obtain (3.5).

For the case when \( m \) is even, the proof is completed by considering (3.4) as for the calculation of \( S^k_{2n-1}(m) \) for odd \( m \). □

3.5. Lemma. For a positive integer \( s \) and a positive even integer \( m \), it holds that

\[
\binom{m+1}{s}_U_k + (-1)^{\frac{m+5}{2}} \binom{m+1}{s-1}_U_k = \frac{U_k(\frac{m+1-s}{2})}{U_k(\frac{m+1}{2})} \binom{m+2}{s}_U_k.
\]

(3.10)

For a positive integer \( s \) and a positive odd integer \( m \), it holds that

\[
\binom{m}{s}_U_k + (-1)^{\frac{m+5}{2}} \binom{m}{s-1}_U_k = \frac{U_k(\frac{m+1-s}{2})}{U_k(\frac{m+1}{2})} \binom{m+1}{s}_U_k.
\]
Proof. For an even positive integer \( m \) and a positive integer \( s \), substituting \( m = \frac{m}{2} - s + 1 \), \( n = \frac{m}{2} + 1 \) in (2.9), we get

\[
U_k(\frac{m}{2} - s + 1) V_k(\frac{m}{2} + 1) = U_k(m - s + 2) + (-1)^{m+s} U_k s.
\]

From (3.11), we get

\[
\left[ \begin{array}{c} \frac{m+1}{s} \\ \end{array} \right] U_k + (-1)^{m+s} \left[ \begin{array}{c} \frac{m+1}{s} \\ \end{array} \right] U_k = \frac{U_k(m+1) U_k m \cdots U_k(m - s + 2)}{(U_k U_{2k} \cdots U_{ks})} + (-1)^{m+s} \frac{U_k(m+1) U_k m \cdots U_k(m - s + 3)}{(U_k U_{2k} \cdots U_{ks+1})}.
\]

Thus the proof is complete. \( \square \)

Define

\[
\sigma_m^k (t) := \sum_{j=0}^{m-t} \left(-1\right)^{\frac{m}{2} (j + m + 1)} \left[ \begin{array}{c} \frac{m+1}{j} \\ \end{array} \right] U_k
\]

where \( m \) is an even positive integer and \( t \) is any integer.

3.6. Lemma. For an even positive integer \( m \) and any integer \( t \),

i) \( \sigma_m^k (t) = 0 \), for \( t \leq -1 \) or \( t \geq m + 1 \),

ii) \( \sigma_m^k (m - t) = \sigma^k (t) \),

iii) \( \sigma_m^k (0) = 1 \), \( \sigma_m^k (1) = 1 + \frac{1}{U_k} (\frac{m+2}{m+1}) U_k (m+1) \),

\[
\sigma_m^k (2) = 1 - \frac{1}{U_k U_{2k}} V_k(\frac{m+2}{m+1}) U_k (m+1) U_k (\frac{m+2}{m+1}),
\]

\[
\sigma_m^k (3) = 1 - \frac{1}{U_k U_{2k} U_{3k}} (\frac{m+2}{m+1}) U_k (m+1) \left( U_k (m+1) V_k(\frac{m+2}{m+1}) U_k (m+1) \right).
\]

Proof. In order to get the proof of i) and ii), one can follow the method of proof of in [12, Lemma 16 i) and ii)], since they are similar statements.

i) The identities for \( \sigma_m^k (0) \) and \( \sigma_m^k (1) \) are directly implied by \( \sigma_m^k (-1) = 0 \). Using ii) and (3.12), we have

\[
\sigma_m^k (2) = \sum_{j=0}^{2} \left(-1\right)^{\frac{m}{2} (j + m + 1)} \left[ \begin{array}{c} \frac{m+1}{j} \\ \end{array} \right] U_k
\]

\[
= 1 + \left(-1\right)^{\frac{m+2}{m+1}} \frac{U_k(m+1) U_k m}{U_k} - \left(-1\right)^{m} \frac{U_k(m+1) U_k m}{U_k U_{2k}}
\]

\[
= 1 - \frac{U_k(m+1)}{U_k U_{2k}} \left( U_k + \left(-1\right)^{\frac{m}{2}} \right)
\]

\[
= 1 - \frac{U_k(m+1)}{U_k U_{2k}} \left( U_k(\frac{m+2}{m+1}) V_k(\frac{m+2}{m+1}) \right).
\]
Lemma 3.7. For an even positive integer $m$ and any integer $t$,

$$\sigma_m^k (t) - \sigma_m^k (t - 2) = (-1)^{\frac{t}{2} (t+m+1)} \left[ \frac{m+2}{t} \right] U_k \left( \frac{m-t+1}{t} \right) \frac{U_k}{U_k (\frac{m-t+1}{t} + 1)}.$$  

Proof. For $t < 2$, the claim follows from the definition of Fibonomial coefficients and Lemma 3.6. For $m \geq 2$, we have

$$\sigma_m^k (t) - \sigma_m^k (t - 2) = \sigma_m^k (m - t) - \sigma_m^k (m - t + 2)$$

$$= \sum_{j=0}^{t} (-1)^{\frac{j}{2} (j+m+1)} \left[ \frac{m+1}{j} \right] U_k - \sum_{j=0}^{t-2} (-1)^{\frac{j}{2} (j+m+1)} \left[ \frac{m+1}{j} \right] U_k$$

$$= (-1)^{\frac{t}{2} (t+m+1)} \left[ \frac{m+1}{t} \right] U_k + (-1)^{\frac{m+1}{t}} \left[ \frac{m+1}{t-1} \right] U_k.$$  

By Lemma 3.5, the proof is complete. □

Lemma 3.8. For an even positive integer $m$ and any integer $t$,

$$\sigma_m^k (t) - \sigma_m^k (t - 4) = (-1)^{\frac{t}{2} (t+m+1)} \left[ \frac{m+4}{t} \right] U_k \frac{U_k}{U_k (\frac{m-t+2}{t} + 1)} \frac{U_k}{U_k (\frac{m-t+2}{t} + 1)} \omega (t, m),$$

where $\omega (t, m) = U_k (\frac{m-t+1}{m+1} - t) V_k (\frac{m-t+2}{m+2} - t) U_k (m+3) - V_k U_k U_k (m+1)$. 

as desired.
Proof. By Lemma 3.7, we have for any integer \( t \) that
\[
\sigma_m^k(t) - \sigma_m^k(t - 4)
= \sigma_m^k(t) - \sigma_m^k(t - 2) + \sigma_m^k(t - 2) - \sigma_m^k(t - 4)

= (-1)^{\frac{1}{2}(t+m+1)} \left[ \frac{m+2}{t} \right] \frac{U_k(\frac{m}{t} - t+1)}{U_k(\frac{m}{t} + 1)}
+ (-1)^{\frac{1}{2}(t+m+1)} \left[ \frac{m+2}{t} \right] \frac{U_k(\frac{m}{t} - t+3)}{U_k(\frac{m}{t} + 1)}

= (-1)^{\frac{1}{2}(t+m+1)} \left[ \frac{m+2}{t} \right] \frac{U_k(\frac{m}{t} - t+1)}{U_k(\frac{m}{t} + 1)}

From (2.10), we get
\[
\sigma_m^k(t) - \sigma_m^k(t - 4)
= (-1)^{\frac{1}{2}(t+m+1)} \left[ \frac{m+2}{t} \right] \frac{U_k(\frac{m}{t} - t+1)}{U_k(\frac{m}{t} + 1)}
\times \left( U_k(\frac{m}{t} - t+1) U_k(m+4-t) U_k(m+3) + U_k U_k(t-1) \right)

\]

as claimed. \( \square \)

3.9. Theorem. For any integer \( m \),
\[
\sum_{j=0}^{t} (-1)^{\frac{1}{2}(j+m+1)} \left[ \frac{m+1}{j} \right] U_k
= \frac{(-1)^{\frac{1}{2}(t+m+1)}}{U_k(\frac{m}{t} + 1) U_k(m+3) U_k(m+4)} \sum_{i=0}^{\frac{1}{2}(t+4)} \left[ \frac{m+4}{t-4i} \right] U_k(\frac{m}{t} - 4i + 1)
\times \left( U_k(\frac{m}{t} + 1) V_k(\frac{m}{t} + 2-t+4) U_k(m+3) - V_k U_k(t-4) U_k(t-4i-1) \right).
\]

Proof. By Lemma 3.8, we can write
\[
\sum_{i=0}^{\frac{1}{4}} \left[ \sigma_m^k(t - 4i) - \sigma_m^k(t - 4(i + 1)) \right] = \sigma_m^k(t) - \sigma_m^k \left( t - 4 \left( \frac{1}{4} \right) + 1 \right).
\]
From Lemma 3.6, we have

\[
\sigma_m^k (t) = \sum_{i=0}^{\lfloor t/4 \rfloor} \left[ \sigma_m^k (t - 4i) - \sigma_m^k (t - 4(i + 1)) \right].
\]

Again by Lemma 3.8, we get

\[
\sigma_m^k (t) = \sum_{i=0}^{\lfloor t/4 \rfloor} (-1)^{t-4i} \left( m+4 \atop t-4i \right) \frac{U_k(m+2-t+4i)}{U_k(m+3)U_k(m+4)} \times \left( U_k(m+1) \right) V_k(m+2-t+4i) U_k(m+3) - V_k U_k(t-4i) U_k(t-4i-1)
\]

\[
= \sum_{i=0}^{\lfloor t/4 \rfloor} (-1)^{t+m+1} \left( m+4 \atop t-4i \right) \frac{U_k(m+2-t+4i)}{U_k(m+3)U_k(m+4)} \times \left( U_k(m+1) \right) V_k(m+2-t+4i) U_k(m+3) - V_k U_k(t-4i) U_k(t-4i-1),
\]

as claimed. \(\square\)

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References